3-D co-geometry

1. Find the point of intersection of the plane $3x - y + 7z + 8 = 0$ and the line

$x = 4 + 5t, y = -2 + t, z = 4 - t$.

Find also the angle between the line and the plane.

Substitute the line in the plane,

$$3(4 + 5t) - (-2 + t) + 7(4 - t) + 8 = 0$$

$$7\,t + 50 = 0$$

$$\therefore t = -\frac{50}{7}$$

$$x = 4 + 5\left(-\frac{50}{7}\right) = -\frac{222}{7}, \qquad y = -2 + \left(-\frac{50}{7}\right) = -\frac{64}{7}, \qquad z = 4 - \left(-\frac{50}{7}\right) = \frac{78}{7}$$

$$\therefore (x, y, z) = \left(-\frac{\mathbf{222}}{\mathbf{7}}, -\frac{\mathbf{64}}{\mathbf{7}}, \frac{\mathbf{78}}{\mathbf{7}}\right)$$

The direction of the given straight line is $\mathbf{r} = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$

The normal of the plane is $\mathbf{N} = 3\mathbf{i} - \mathbf{j} + 7\mathbf{k}$

$$\mathbf{r} \cdot \mathbf{N} = |\mathbf{r}||\mathbf{N}|\cos(\mathbf{r}, \mathbf{N})$$

$$\Longrightarrow (5)(3) + (1)(-1) + (-1)(7) = \sqrt{5^2 + 1^2 + (-1)^2}\sqrt{3^2 + (-1)^2 + 7^2}\cos(\mathbf{r}, \mathbf{N})$$

$$\Longrightarrow \cos(\mathbf{r}, \mathbf{N}) = \frac{7\sqrt{19}}{171}$$

Angle between the line and the plane $= \theta = \frac{\pi}{2} - \angle(\mathbf{r}, \mathbf{N}) \Longrightarrow \cos(\mathbf{r}, \mathbf{N}) = \sin\theta$

$$\therefore \sin\theta = \frac{7\sqrt{19}}{171} \Longrightarrow \quad \theta = \mathbf{10.2785847750013}° \ \ (\text{or } \mathbf{0.1793951467691\ rad.})$$

2. The position vectors of points P and Q referred to an origin O are given by $\mathbf{OP} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{OQ} = 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$ respectively. Show that the cosine of $\angle POQ$ is equal to $\frac{4}{\sqrt{38}}$.

Hence, or otherwise, find the position vector of M on OQ such that PM is perpendicular to OQ.

$$\mathbf{OP} \cdot \mathbf{OQ} = |\mathbf{OP}||\mathbf{OQ}| \cos\angle POQ$$

$$(3\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) = |3\mathbf{i} + \mathbf{j} + 3\mathbf{k}||5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}| \cos\angle POQ$$

$$(3)(5) + (1)(-4) + (3)(3) = \sqrt{3^2 + 1^2 + 3^2}\sqrt{5^2 + (-4)^2 + 3^2} \cos\angle POQ$$

$$20 = \sqrt{19}\sqrt{50} \cos\angle POQ$$

$$\therefore \cos\angle POQ = \frac{20}{\sqrt{19}\sqrt{50}} = \frac{\mathbf{4}}{\sqrt{\mathbf{38}}}$$

Since M on $\mathbf{OQ}$, $\mathbf{OM} = m(5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) = 5m\mathbf{i} - 4m\mathbf{j} + 3m\mathbf{k}$, where m is a scalar.

$\mathbf{PM} = \mathbf{OM} - \mathbf{OP} = (5m - 3)\mathbf{i} + (-4m - 1)\mathbf{j} + (3m - 3)\mathbf{k}$

Since $\mathbf{PM} \perp \mathbf{OQ}$, $\mathbf{PM} \cdot \mathbf{OQ} = 0$

$(5m - 3)(5) + (-4m - 1)(-4) + (3m - 3)(3) = 0$

$$\therefore m = \frac{2}{5}$$

$$\therefore \mathbf{OM} = \frac{2}{5}(5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) = 2\mathbf{i} - \frac{8}{5}\mathbf{j} + \frac{6}{5}\mathbf{k}$$

3. Given two lines $l_1: x = 1 + 2t,\ y = -2 + 3t,\ z = 2 + 2t$

$l_2: \frac{1-x}{2} = \frac{y-2}{3} = z - 1$

Determine whether these lines are parallel, intersect or skewed.

Rewrite l_2 in parametric form,

$l_2: x = 1 - 2t,\ y = 2 + 3t,\ z = 1 + t$

Direction of l_1: $\mathbf{r_1} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

Direction of l_2: $\mathbf{r_2} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

Since $\mathbf{r_1} \neq k\mathbf{r_2}$ for any scalar k, l_1 and l_2 are not parallel.

Equate the corresponding x, y z values,

$$\begin{cases} 1 + 2t = 1 - 2t \\ -2 + 3t = 2 + 3t \\ 2 + 2t = 1 + t \end{cases} \Rightarrow \begin{cases} t = \frac{1}{4} \\ \text{no solution for t} \\ t = -1 \end{cases} \Rightarrow \text{no solution for t}$$

There is no intersection point for l_1 and l_2.

$\therefore$ l_1 and l_2 are skewed.

4. $\pi_1: 2x - y + z + 5 = 0$, $\pi_2: 4x + y + z - 7 = 0$ are two planes.

O is the origin and the point P has coordinates (4,5,2).

(a) Verify that the vector $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ is parallel to both π_1 and π_2.

(b) Find the vector equation of the plane which passes through P and is perpendicular to both π_1 and π_2.

(c) Find the coordinates of one point common to π_1 and π_2 and hence, find the Cartesian equation of the line of intersection of π_1 and π_2.

(a) The normal to the plane $\pi_1: 2x - y + z + 5 = 0$ is $\mathbf{N_1} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

The normal to the plane $\pi_2: 4x + y + z - 7 = 0$ is $\mathbf{N_2} = 4\mathbf{i} + \mathbf{j} + k$.

$\mathbf{r} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

Since $\mathbf{r} \cdot \mathbf{N_1} = (-1)(2) + (1)(-1) + (3)(1) = 0$

$\mathbf{r} \cdot \mathbf{N_2} = (-1)(4) + (1)(1) + (3)(1) = 0$

$\therefore$ The vector $-\mathbf{i}+\mathbf{j}+3\mathbf{k}$ is parallel to both π_1 and π_2.

(b) $\mathbf{n}=\mathbf{N_1}\times\mathbf{N_2}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 1\\ 4 & 1 & 1\end{vmatrix}=-2\mathbf{i}+2\mathbf{j}+6\mathbf{k}$

Since $\mathbf{N_1}\times\mathbf{N_2}$ is a vector perpendicular to the plane made by the normals of π_1 and π_2, $\mathbf{n}$ is the normal of the plane perpendicular to both π_1 and π_2.

Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

Let the vector equation of the plane which passes through P and is perpendicular to both π_1 and π_2 is π and $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ be a variable point on π.

The required vector equation is

$$\pi:\ (\mathbf{r}-\mathbf{OP})\cdot\mathbf{n}=0$$

$$[(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})-(4\mathbf{i}+5\mathbf{j}+2\mathbf{k})]\cdot(-2\mathbf{i}+2\mathbf{j}+6\mathbf{k})=0$$

(c) We can find a common point by putting $x=0$ in π_1 and π_2.

We get $\begin{cases}-y+z+5=0\\ y+z-7=0\end{cases}$.

Solve we get $y=6, z=1$

Then $P=(0,6,1)$ is a point common to π_1 and π_2.

$\mathbf{n}=\mathbf{N_1}\times\mathbf{N_2}=-2\mathbf{i}+2\mathbf{j}+6k$ in (b) is a vector parallel to the line of intersection of $\pi_1$and π_2.

The required equation is $\frac{x-0}{-2}=\frac{y-6}{2}=\frac{z-1}{6}$.

5. The points A, B, C and D have positive vectors, relative to the origin O, given by $\mathbf{OA}=\mathbf{i}+a\mathbf{j}-\mathbf{k}$, $\mathbf{OB}=-\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $\mathbf{OC}=2\mathbf{i}+\mathbf{j}+4\mathbf{k}$ and $\mathbf{OD}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, where a is a constant.
Given OA is perpendicular to OB,

(a) find the value of a and **OA**,

(b) show that **OA** is normal to the plane OBC,

(c) find an equation of the plane through D parallel to the plane OBC and, hence, find the position vector of the point of intersection of this plane and the line AC.

(a) $\mathbf{OA}\cdot\mathbf{OB}=0$, $(1)(-1)+(a)(2)+(-1)(3)=0$, $a=2$

$\mathbf{OA}=i+2j-k$

(b) $\mathbf{N}=\mathbf{OB}\times\mathbf{OC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 2 & 3\\ 2 & 1 & 4\end{vmatrix}=5\mathbf{i}+10\mathbf{j}-5\mathbf{k}$ which is the normal of the plane OBC.

Since $\mathbf{N}=5\mathbf{i}+10\mathbf{j}-5\mathbf{k}=2(i+2\mathbf{j}-\mathbf{k})=5\,\mathbf{OA}$

Therefore OA is normal to the plane OBC.

(c) Since O is on the plane, OBC: $5x+10y-5z=0$.

Let the required plane be $\pi: 5x+10y-5z=k$

D is on π, therefore $5(1)+10(1)-5(1)=k$.
Solve we get $k=10$.
$\pi: 5x+10y-5z=10$ or $\pi: x+2y-z=2$

$\mathbf{AC}=\mathbf{OC}-\mathbf{OA}=(2\mathbf{i}+\mathbf{j}+4\mathbf{k})-(\mathbf{i}+2\mathbf{j}-\mathbf{k})=\mathbf{i}-\mathbf{j}-5\mathbf{k}$
The line AC must pass through $A(1,2,-1)$.
$AC: x=1+t, y=2-t, z=-1-5t$
It cuts the plane $\pi: x+2y-z=2$, therefore $(1+t)+2(2-t)-(-1-5t)=2$

$$t=-1$$

$\therefore$ Intersection point $(x,y,z)=(0,3,4)$

6. Let the equations of two planes be π_1: $\mathbf{r}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}=2$ and π_2: $\mathbf{r}\cdot\begin{pmatrix}1\\-3\\4\end{pmatrix}=3$.

(a) Find the acute angle between π_1 and π_2, giving your answer to the nearest $0.1°$.
(b) Determine the length of the projection of the vector $\mathbf{i}+3\mathbf{j}$ to π_1 .
(c) Find the equation of the plane π_3 which is perpendicular to both π_1 and π_2 and passes though the point $P(1,3,-2)$.

(a) The angle between two intersecting planes is the angle between their normal vectors.

$\mathbf{N_1}=\begin{pmatrix}1\\1\\2\end{pmatrix}$, $\mathbf{N_2}=\begin{pmatrix}1\\-3\\4\end{pmatrix}$ are the normals of π_1 and π_2 respectively.

$\mathbf{N_1}\cdot\mathbf{N_2}=|\mathbf{N_1}||\mathbf{N_2}|\cos\theta$
$(1)(1)+(1)(-3)+(2)(4)=\sqrt{1^2+1^2+2^2}\sqrt{1^2+(-3)^2+4^2}\cos\theta$

$$\cos\theta=\frac{(1)(1)+(1)(-3)+(2)(4)}{\sqrt{1^2+1^2+2^2}\sqrt{1^2+(-3)^2+4^2}}=\frac{\sqrt{39}}{13}$$

$\theta=\mathbf{61.3°}$, to the nearest $0.1°$.

(b) Let $\mathbf{r}=\mathbf{i}+3\mathbf{j}$, $\mathbf{r}\cdot\mathbf{N_1}=|\mathbf{r}||\mathbf{N_1}|\cos\angle(\mathrm{r},\mathrm{N}_1)$

$$(1)(1)+(3)(1)+(0)(2)=\sqrt{1^2+3^2+0^2}\sqrt{1^2+2^2+2^2}\cos\angle(\mathbf{r},\mathbf{N_1})$$
$$4=3\sqrt{10}\cos\angle(\mathrm{r},\mathrm{N}_1)$$
$$\cos\angle(\mathrm{r},\mathrm{N}_1)=\frac{4}{3\sqrt{10}}=\frac{2\sqrt{10}}{15}$$
$$\sin\angle(\mathrm{r},\mathrm{N}_1)=\sqrt{1-\left(\frac{2\sqrt{10}}{15}\right)^2}=\frac{\sqrt{185}}{15}$$

$\therefore$ Length of the projection of the vector $\mathrm{i}+3\mathrm{j}$ to $\pi_1=|\mathrm{r}|\sin\angle(\mathrm{r},\mathrm{N}_1)=\sqrt{10}\times\frac{\sqrt{185}}{15}=\frac{\sqrt{74}}{3}$

(c) $\mathbf{n} = \mathbf{N_1} \times \mathbf{N_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 1 & -3 & 4 \end{vmatrix} = 10\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$

Since $\mathbf{N_1} \times \mathbf{N_2}$ is a vector perpendicular to the plane made by the normals of π_1 and π_2, $\mathbf{n}$ is the normal of the plane perpendicular to both π_1 and π_2.
Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
Let the vector equation of the plane which passes through P and is perpendicular to both π_1 and π_2 is π and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be a variable point on π.
The required vector equation is

$$\pi: \ (\mathbf{r} - \mathbf{OP}) \cdot \mathbf{n} = 0$$
$$[(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) - (\mathbf{i} + 3\mathbf{j} - 2k)] \cdot (10\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}) = 0$$
$$(x-1)(10) + (y-3)(-4) + (z+2)(-2) = 0$$
$$5x - 2y - z - 1 = 0$$

7. $A(1,4,2), B(3,-1,5), C(2,6,0), D(4,3,-1) E(3,7,3)$ and $F(4,5,2)$ are six points in three dimensional space. The points A, B and C lie on the plane π_1, whereas the points D, E and F lie on the plane π_2. A straight line L_1 passes through the points E and F.

(a) Determine whether $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular vectors.

(b) Find the Cartesian equation of the plane π_1.

(c) Find the equation of line L_1 in vector form and in parametric form.

(d) Find the coordinates of the point of intersection of the line L_1 and the plane π_1.

(e) Find the Cartesian equation of the straight line L_2 passes through the point $(10,8,9)$ and perpendicular to the plane π_1 .

(f) Find the position vector of the point of intersection between the lines L_1 and L_2 .

(g) Find the acute angle between the plane π_1 and the plane π_2.

(a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}, \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.
$\overrightarrow{AB} \cdot \overrightarrow{AC} = (2)(1) + (-5)(2) + (3)(-2) = -14 \neq 0$
$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not perpendicular vectors.

(b) $\overrightarrow{AC} \times \overrightarrow{AC} = \mathbf{N_1} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 3 \\ 1 & 2 & -2 \end{vmatrix} = 4\mathbf{i} + 7\mathbf{j} + 9\mathbf{k}.$ which is the normal of the plane π_1.

Since $A(1,4,2)$ is on the plane, the Cartesian equation of the plane π_1 is

$4(x-1)+7(y-4)+9(z-2)=0$

$\pi_1: 4x+7y+9z=50$

(c) $\overrightarrow{EF}=\overrightarrow{OF}-\overrightarrow{OE}=\mathbf{i}-2\mathbf{j}-\mathbf{k}$

Since $E(3,7,3)$ is on L_1, equation of the line L_1 is

$\vec{\mathbf{r}}=3\mathbf{i}+7\mathbf{j}+3\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{j}-\mathbf{k})$

In parametric form, $L_1:\begin{cases} x=3+\lambda \\ y=7-2\lambda \\ z=3-\lambda \end{cases}$

(d) Put L_1 in π_1,

$$4(3+\lambda)+7(7-2\lambda)+9(3-\lambda)=50$$

$$\lambda=2$$

The point of intersection of the line L_1 and the plane π_1 is

$x=3+\lambda=5, y=7-2\lambda=3, z=3-\lambda=1$

or $(5,3,1)$

(e) The normal of the plane π_1 is $\mathbf{N}=4\mathbf{i}+7\mathbf{j}+9\mathbf{k}$ which is the direction of L_2.

The Cartesian equation of the straight line L_2 passes through the point $(10,8,9)$ and perpendicular to the plane π_1 is

$L_2: \frac{x-10}{4}=\frac{y-8}{7}=\frac{z-9}{9}$

(f) Substitute L_1 in L_2, we get $\frac{(3+\lambda)-10}{4}=\frac{(7-2\lambda)-8}{7}=\frac{(3-\lambda)-9}{9}$

$$\therefore \lambda=3$$

The required point is $x=3+\lambda=6, y=7-2\lambda=1, z=3-\lambda=0$.

The position vector of the point of intersection between the lines L_1 and L_2 is

$6\mathbf{i}+\mathbf{j}$

(g) $D(4,3,-1)E(3,7,3)$ and $F(4,5,2)$

$$\overrightarrow{DE}\times\overrightarrow{DF}=\mathbf{N_2}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -3 & -1\\ 1 & 1 & 3\end{vmatrix}=-8\mathbf{i}-7\mathbf{j}+5\mathbf{k}$$

$\mathbf{N_1}\cdot\mathbf{N_2}=(4\mathbf{i}+7\mathbf{j}+9\mathbf{k})\cdot(-8\mathbf{i}-7\mathbf{j}+5\mathbf{k})=(4)(-8)+(7)(-7)+(9)(5)=-36$

$$\mathbf{N_1}\cdot\mathbf{N_2}=|\mathbf{N_1}||\mathbf{N_2}|\cos(\mathbf{N_1},\mathbf{N_2})=\sqrt{4^2+7^2+9^2}\sqrt{(-8)^2+(-7)^2+5^2}\cos(\mathbf{N_1},\mathbf{N_2})$$

$$=2\sqrt{5037}\cos(\mathbf{N_1},\mathbf{N_2})$$

$2\sqrt{5037}\cos(\mathbf{N_1},\mathbf{N_2})=-36$

$\cos(\mathbf{N_1},\mathbf{N_2})=\frac{-36}{2\sqrt{5037}}$

$\angle(\mathbf{N_1},\mathbf{N_2})=104.7°$

$\therefore$ The acute angle between the plane π_1 and the plane π_2 is $180°-104.7°=\mathbf{75.3°}$

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